# Sheaves on Proj

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Let $R = \bigoplus_{j \in \mathbb{N}} R_j$ be an $\mathbb{N}$-graded ring. We constructed a scheme $(X, \mathcal{O}_X) = (\mathrm{Proj}(R), \mathcal{O}_{\mathrm{Proj}(R)})$; Set $R_+ = \bigoplus_{j>0} R_j$

Recall $\mathrm{Proj}(R) = \{ p \mid p$ homogeneous prime, $p \not\supseteq R_+ \}$

The top on $\mathrm{Proj}(R)$ has open basis $\{D_+(f)\}$ f homo of +ve deg

$D_+(f) = \{ p \in \mathrm{Proj}(R) \mid f \notin p \}$

$\mathcal{O}_X(D_+(f)) \cong (R[1/f])_0$.

§ Recall, a $\mathbb{Z}$-graded $R$ mod is $M \in \mathrm{Mod}_R$ such that $M = \bigoplus_{\lambda \in \mathbb{Z}} M_\lambda$ as an ab group and $R_{\lambda_1} \cdot M_{\lambda_2} \subseteq M_{\lambda_1+\lambda_2}$, $\forall \lambda_1 \in \mathbb{N}, \lambda_2 \in \mathbb{Z}$.

- A graded $R$-lin map (or simply a graded map) between two graded mods is an $R$ lin map $\varphi: M \to N$ s.t $\varphi(M_\lambda) \subseteq N_\lambda$ $\forall \lambda \in \mathbb{Z}$.
- The cat of $\mathbb{Z}$-graded $R$-mods with graded $R$-lin map forms a cat denoted $\mathrm{Mod}_R^{gr}$

Given $M \in \mathrm{Mod}_R^{gr}$, we construct a sheaf of $\mathcal{O}_X$-mods, denoted $\tilde{M}$ on $X = \mathrm{Proj}\ R$

Caution: This $\tilde{M}$ is not the qcoh sheaf $\tilde{M}$ on $\mathrm{Spec}(A)$.

- Note given a mult closed set $S$ of homogeneous elts of $R$, $S^{-1}R$ is $\mathbb{Z}$-graded ring. $S^{-1}M$ is a $\mathbb{Z}$-graded mod / $S^{-1}R$.

$$(S^{-1}R)_\lambda = \{ r/s \in S^{-1}R \mid r, s \text{ homo}, \deg r - \deg s = \lambda \}$$

$$(S^{-1}M)_\lambda = \{ m/s \in S^{-1}M \mid m, s \text{ homo}, \deg m - \deg s = \lambda \}$$

- For $p \in \mathrm{Proj}(R)$, set $S_p$ = homo elts of $R \setminus p$

$$M_{(p)} = (S_p^{-1} M)_0$$

**Def of $\tilde{M}$:** Consider the fun sheaf of $\mathcal{O}_X$-mods on $X$;

$$\tilde{M}(U) = \Big\{ \text{set maps } s: U \to \coprod_{p \in U} M_{(p)} \ \Big|\ \begin{array}{l} \bullet\ s(p) \in M_{(p)} \\ \bullet \text{ For every } U, \exists \text{ a open nbhd } V \subseteq U \text{ of } p, \text{ a homo } f \notin \bigcup_{q \in V} q \\ \quad \& \ m \in M \text{ s.t } s(q) = m/f \in M_{(q)} \end{array} \Big\}$$

It's implicit that $\deg m = \deg f$

Recall: $\tilde{R} = \mathcal{O}_X$, so for $s \in \mathcal{O}_X(U)$, $t \in \tilde{M}(U)$

$$(s\,t)(q) = \underset{R_{(q)}}{s(q)} \cdot \underset{M_{(q)}}{t(q)} \in M_{(q)} \quad \forall q \in U$$

Thm: 1) $\tilde{M}$ is a sheaf for any $M \in \mathrm{Mod}_R^{gr}$.
For $p \in \mathrm{Proj}\,R$, the natural map $M_{(p)} \to (\tilde{M})_p$ is an isom with inverse given by $s \mapsto s(p)$ for a section $s \in \tilde{M}(U)$, $p \in U$.

2) Given a graded $R$-lin map $\varphi: M \to N$, naturally have $\tilde{\varphi}: \tilde{M} \to \tilde{N}$

for $t \in \tilde{M}(U)$, $p \in U$: $\tilde{\varphi}_U(t)(p) = \varphi_{(p)}(t(p))$; $\varphi_{(p)}: M_{(p)} \to N_{(p)}$

3) $\widetilde{\varphi_1 \cdot \varphi_2} = \tilde{\varphi}_1 \cdot \tilde{\varphi}_2$, $\widetilde{id_M} = id_{\tilde{M}}$

4) So $M \to \tilde{M}$ is gives a functor $\mathrm{Mod}_R^{gr} \to \mathrm{Mod}_{\mathcal{O}_X}$.

Thm: For $M \in \mathrm{Mod}_R^{gr}$, $\tilde{M} \in \mathrm{Qcoh}(X)$.
Moreover $\Gamma(D_+(f), \tilde{M}) \cong (M[1/f])_0$ for

**Thm.** For $M \in Mod_R^{gr}$, $\tilde{M} \in Qcoh(X)$. Moreover $\Gamma(D_+(f), \tilde{M}) \cong (M[1/f])_0$ for every homo elt $f$ of +ve deg.

**Pf.** Since being qcoh is a local property, it's enough to check that for any homo elt $f$ of of +ve deg $\tilde{M}|_{D_+(f)}$ is qcoh.

Set $\Gamma = M[1/f]_0$. The identity map $\Gamma \to \Gamma(D_+(f), \tilde{M})$ gives an $\mathcal{O}_{D_+(f)}$-line map

$*$ $\tilde{\Gamma} \longrightarrow \tilde{M}|_{D_+(f)}$ as $\tilde{\Gamma}$ is qcoh on the affine $D_+(f)$.

**Claim.** $*$ is an isom.

**Pf.** We check isom at the stalks at $p \in D_+(f)$.

Recall that $D_+(f) \xrightarrow{\sim} Spec(R[1/f]_0)$

$q \longmapsto qR[1/f] \cap R[1/f]_0 = q_a$

The map induced by $* : (M[1/f]_0)_{p_a} = \tilde{\Gamma}_{p_a} \longrightarrow \tilde{M}_p \xrightarrow{\sim} M_{(p)}$ $**$

$$g \in S_p, \quad \frac{m/f^a}{g/f^{e_2}} \longmapsto \frac{m f^{e_2}}{g f^{a}}$$

Note $(M[1/f]_0)_{p_a} = [(S_p)^{-1} M[1/f]]_0 = (S_p^{-1}M)_0$

So $**$ is an isom.

**Rmk.** So $\tilde{M}$ can be thought to be obtained by gluing $\widetilde{M[1/f]_0}$ on $D_+(f)$ for different $f$'s.

**Thm.** 1) $f: M \to N$ be a graded $R$-mod map. $\ker \tilde{f} \xleftarrow{\sim} \widetilde{\ker f}$, $\operatorname{coker} \tilde{f} \xrightarrow{\sim} \widetilde{\operatorname{coker} f}$.

2) If $0 \to M' \to M \to M'' \to 0$ is exact in $Mod_R^{gr}$ [This includes the hypothesis that all the maps are graded] Then $0 \to \tilde{M}' \to \tilde{M} \to \tilde{M}'' \to 0$ is also exact in $Mod_{\mathcal{O}_X}$

3) $(\bigoplus_{i \in I} M_i)^{\sim} \xleftarrow{\sim} \bigoplus_{i \in I} \tilde{M}_i$ in $Mod_{\mathcal{O}_X}$ for any set $I$.

**Thm.** 1) $\tilde{M} = 0 \iff \forall f$ homo of +ve deg $M[1/f]_0 = 0$
$\iff$ for a collection of homo elt of +ve deg $f_1, f_2, \dots, f_n$ s.t $X = \bigcup_{i=1}^{n} D_+(f_i)$ and and any $m \in M_n$ where $\deg f_i \mid n \ \forall i$, $\exists n_1, n_2, \dots n_n$ s.t $f_i^{n_i} \cdot m = 0$

[A special case is when one can choose $f_i$'s of deg 1 s.t $X = \bigcup_{i=1}^{n} D(f_i)$]

2) For $M \in Mod_R^{gr}$, $\tilde{i}$ is an isom.

$i: \bigoplus_{\lambda \in \mathbb{N}} M_\lambda \hookrightarrow M$.

**Pf.** $\tilde{M} = 0 \iff \forall f$ homo of +ve deg $\tilde{M}|_{D_+(f)} = 0$
$\iff$ ———————" ——— $M[1/f]_0 = 0$
($\tilde{M}$ qcoh)
$\iff$ for a collection $\{f_i\}_{i \in I}$, $f_i$ homo of +ve deg s.t $X = D_+(f_i)$, $M[1/f_i]_0 = 0$

Now assume $X = \bigcup_{i=1}^{n} D(f_i)$

s.t $X = D_+(f_i)$, $M[1/f_i]_0$

Now assume $X = \bigcup_{i=1}^{r} D(f_i)$

$M[1/f_i]_0 = \{ m/f_i^{n} \mid \deg m = n \deg f_i \}$

$M[1/f_i]_0 = 0 \iff$ for $m \in M_\lambda$ s.t $\deg f_i \mid \lambda$

$m/f_i^{\lambda/\deg f_i} = 0 \in M[1/f_i]_0 \subseteq M[1/f_i]$

$\iff f_i^{n_i} m = 0$ for some $n_i$.

2) Consider the exact seq in $Mod_R^{gr}$

$$0 \to \bigoplus_{\lambda \in \mathbb{N}} M_\lambda \to \underbrace{\bigoplus_{\lambda \in \mathbb{Z}} M_\lambda}_{M} \to Q \to 0$$

Every nonzero homo elt in $Q$ lifts to a non-zero homo elt, $m$, of $-$ve deg in $M$

Now for every homo elt, $f$, of $R$ of $+$ve deg

$\deg f^n . m \geq 0$ for $n >> 0 \implies f^n . \bar{m} = 0$ ($\bar{m}$: im in $Q$)

$\implies \tilde{Q} = 0$

$\implies 0 \to \widetilde{\bigoplus_{\lambda \in \mathbb{N}} M_\lambda} \to \tilde{M} \to 0$ is exact.

End of 08.11.24 Lec

§. $\mathcal{O}_X(n)$, $\mathcal{F}(n)$, ... $X = (\operatorname{Proj}(R), \mathcal{O}_{\operatorname{Proj}(R)})$

**Def.** • For $M \in Mod_R^{gr}$, $M(n) \in Mod_R^{gr}$ is the object whose underlying $R$-mod is $M$, but

$$M(n)_m = M_{n+m}$$

• $\mathcal{O}_X(n) := \widetilde{R(n)}$

• For $\mathcal{F} \in Mod_{\mathcal{O}_X}$, $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$

$\mathcal{F}(n)$ is called the $n$-th Serre twist of $\mathcal{F}$.

Note There is a map $M_0 \to \Gamma(X, \tilde{M})$ and so

$M_n = (M(n))_0 \longrightarrow \Gamma(X, \widetilde{M(n)})$

**Def:** $Y$ scheme, $\mathcal{G} \in Mod_{\mathcal{O}_Y}$ is called loc. free if $\exists$ an open covering $Y = \bigcup_{\alpha \in I} U_\alpha$ s.t $\mathcal{F}|_{U_\alpha} \xrightarrow{\sim} \bigoplus^{r_\alpha} \mathcal{O}_{U_\alpha}$, $r_\alpha \in \mathbb{N}$ in $Mod_{\mathcal{O}_{U_\alpha}}$

**Prop** • There are natural maps $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \to \mathcal{O}_X(n+m)$ and $\mathcal{F}(n) \otimes \mathcal{O}_X(m) \to \mathcal{F}(n+m)$.

• Suppose $\deg f = m > 0$, Then

(i) $\mathcal{O}_{D_+(f)} \longrightarrow \mathcal{O}_X(m)|_{D_+(f)}$ is an isom in $Mod_{\mathcal{O}_{D_+(f)}}$

$1 \longmapsto f/1$

(ii) $\mathcal{F}(n)|_{D_+(f)} \otimes \mathcal{O}_X(m)|_{D_+(f)} \xrightarrow{\sim} \mathcal{O}_X(n+m)|_{D_+(f)}$

Pf. (i) Using qcoh, enough to prove

$$\begin{array}{ccc} \Gamma(D_+(f), \mathcal{O}_X) & \longrightarrow & \Gamma(D_+(f), \mathcal{O}_X(m)) \\ \| & & \uparrow\wr \\ R[1/f]_0 & \longrightarrow & [R(m)[1/f]]_0 = (R[1/f])_m \\ 1 & \longmapsto & f/1 \end{array}$$

is an isom

$R[1/f]_0 \longrightarrow \Gamma(D_+(f), \mathcal{O}_X)$

$1 \longmapsto f/1$

is an isom

inverse $s/f \longleftarrow s$

(ii) Since $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_X(n)$, enough to prove

$\mathcal{O}_X(n)|_{D_+(f)} \otimes \mathcal{O}_X(m)|_{D_+(f)} \longrightarrow \mathcal{O}_X(n+m)|_{D_+(f)}$

is an isom

q-coh $\Rightarrow$ enough to check the isom

$x \otimes y \longmapsto xy$

$$R[1/f]_n \otimes_{R[1/f]_0} R[1/f]_m \longrightarrow R[1/f]_{n+m}$$

$$\downarrow\wr \qquad\qquad\qquad\qquad \|$$

$$R[1/f]_n \otimes_{R[1/f]_0} f \cdot R[1/f]_0 \xrightarrow{\sim} f \cdot R[1/f]_n$$

§ Given $\mathcal{F} \in QCoh(X)$, $X = \mathrm{Proj}(R)$ we want to construct a <u>candidate</u> mod $M$ s.t. $\tilde{M} \xrightarrow{\sim} \mathcal{F}$.

<u>Caution.</u> Such an $M$ need not exist.

<u>Def.</u> Given $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$, set $\Gamma_* \mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$

$\underline{\Gamma_* \mathcal{F}} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}(n)$

<u>Prop</u>
- $\Gamma_* \mathcal{F}$ is naturally a $\Gamma_* \mathcal{O}_X$ mod
- There is a map of graded rings $R \to \Gamma_* \mathcal{O}_X$.
- Thus $\Gamma_* \mathcal{F}$ is naturally an $R$-mod.
- There is an $\mathcal{O}_X$-lin map $\widetilde{\Gamma_* \mathcal{F}} \longrightarrow \mathcal{F}$

<u>Pf</u>
- $\Gamma_* \mathcal{O}_X = \bigoplus_{n \in \mathbb{N}} \Gamma(X, \mathcal{O}_X(n))$, have $\Gamma_X(X, \mathcal{O}_X(m)) \times \Gamma_X(X, \mathcal{O}_X(n))$

$\downarrow$

$\Gamma(X, \mathcal{O}_X(n+m))$

making $\Gamma_* \mathcal{O}_X$ a ring.

Have $\Gamma(X, \mathcal{F}(n)) \times \Gamma(X, \mathcal{O}_X(m)) \longrightarrow \Gamma(X, \mathcal{F}(n+m))$.

$\| \qquad\qquad \| \qquad\qquad \|$

$\Gamma_* \mathcal{F}_m \qquad \Gamma_* \mathcal{O}_{X\,m} \qquad \Gamma_* \mathcal{F}_{m+n}$

- $R_n \longrightarrow \Gamma(X, \mathcal{O}_X(n))$

$r \longmapsto r/1$

- We describe $\widetilde{\Gamma_* \mathcal{F}}(D_+(f)) \longrightarrow \mathcal{F}(D_+(f))$

q coh $\Rightarrow$ Enough to produce $\Gamma_* \mathcal{F}[1/f]_0 \longrightarrow \Gamma(D_+(f), \mathcal{F})$

Note $\Gamma_* \mathcal{F}|_{D_+(f)} = \bigoplus_{\lambda \in \mathbb{Z}} \Gamma(D_+(f), \mathcal{F}(\lambda))$ is

not a $\Gamma_* \mathcal{O}_{D_+(f)} = \bigoplus_{\lambda \in \mathbb{Z}} \Gamma(D_+(f), \mathcal{O}_X(\lambda m))$ mod.

In the later ring the image of $f$ via

$R \longrightarrow \Gamma_* \mathcal{O}_X$ is invertible as

$\searrow \qquad \downarrow \qquad 1/f \in \Gamma(D_+(f), \mathcal{O}(-\deg f))$

graded $\Gamma_* \mathcal{O}_{D_+(f)}$

$\frac{1}{f} \in \Gamma(D_+(f), \mathcal{O}(-\deg f))$

$\Gamma_* \mathcal{O}_{D_+(f)}$

Thus have a graded $R[1/f] \longrightarrow \Gamma_* \mathcal{O}_{D_+(f)}$. This makes $\Gamma_* \mathcal{F}|_{D_+(f)}$ an $R[1/f]$ module and induces a graded $R$-lin map, $\Gamma_* \mathcal{F}[1/f] \longrightarrow \Gamma_* \mathcal{F}|_{D_+(f)}$

Restrict to the deg 0 piece to obtain the desired map.

<u>Thm</u> Assume $R$ is a finitely generated $R_0$ alg; i.e $\exists$ $g_1, g_2 \ldots g_r$ +ve deg homo elts s.t $R_0[x_1, \ldots x_r] \longrightarrow R$, $x_i \longmapsto g_i$, $R_0$-lin is sur.

- For $\mathcal{F} \in Qcoh(X)$, the map $\widetilde{\Gamma_* \mathcal{F}} \longrightarrow \mathcal{F}$ is an isom.

<u>Pf:</u> Our hypothesis $\Rightarrow$ $R_+ = \sqrt{(g_1, g_2 \ldots g_r)}$ ideal

$\Rightarrow X = \bigcup_{i=1}^{r} D_+(g_i)$

Replace $g_i$ by $g_i^{\prod_{j \neq i} \deg g_j}$ to assume $\deg g_i$ are all equal $= d$. Note $X = \bigcup_{i=1}^{r} D_+(g_i)$.

Note $\mathcal{O}_X(nd)$ is invertible for $\forall n$.

Since $\mathcal{F}$ is qcoh it is enough to check that the induced map $(\Gamma_* \mathcal{F}[1/g_i])_0 \longrightarrow \Gamma(D_+(g_i), \mathcal{F})$ $(+)$ is an isom $\forall i$

<u>Needed lemma</u>

<u>Lemma:</u> $Y$ quasi compact scheme, $\mathcal{G} \in Qcoh(Y)$, $\mathcal{L}$ invertible, $s \in \Gamma(Y, \mathcal{L})$. Let

$D_s = \{ y \in Y \mid s_y \notin m_y \mathcal{L}_y \}$ $m_y$ is the max ideal of $\mathcal{O}_{Y,y}$.

(i) $D_s$ is open

(ii) Given $t_1 \in \Gamma(Y, \mathcal{G})$ if $t_1|_{D_s} = 0$, $\exists$ $n \in \mathbb{N}$ s.t $t_1 \otimes s^n = 0 \in \Gamma(Y, \mathcal{G} \otimes \mathcal{L}^n)$

(iii) Assume intersection of every two open affines have a finite covering by affine opens in $Y$. [This is immediate if $Y$ is (quasi) separated]

(iii) Assume intersection of every ... covering by affine opens in $Y$. [This is immediate if $Y$ is noetherian or $Y$ is (quasi) separated]

Given $t \in \Gamma(D_s, \mathcal{G})$, $\exists\ n \in \mathbb{N}$, $\tilde{t} \in \Gamma(Y, \mathcal{G} \otimes \mathcal{L}^n)$

s.t $t \otimes s^n = \tilde{t}|_{D_s}$

$\quad \in \Gamma(D_s, \mathcal{G} \otimes \mathcal{L}^n)$

Pf. (i) [Quasi compactness is unnecessary], Choose an affine open covering $Y = \bigcup_j U_j$ s.t

$\mathcal{O}_{U_j} \xrightarrow{\sim} \mathcal{L}|_{U_j}$ $\quad 1 \longrightarrow s_j$, $s|_{U_j} = f_j s_j$ $\quad f_j \in \Gamma(U_j, \mathcal{O})$ )

$D_s \cap U_j = D(f_j) \underset{\text{open}}{\subseteq} U_j$. Thus $D_s \cap U_j$ is open in $U_j\ \forall j$. Thus $D_s$ is open in $X$. ◻

Since $Y$ is quasi-compact, we choose a finite subcover of $U_j$ $\quad Y = \bigcup_{j=1}^{r} U_j$. Fix this for (ii), (iii)

(ii) $t_1|_{U_j \cap D_s} = 0$, $U_j \cap D_s = D_{U_j}(f_j)$ [This means the basic affine open given by $f_j$ inside $U_j$]

Since $U_j$ is affine and $\mathcal{G}|_{U_j}$ is qcoh $\quad f_j^{n_j} \cdot t_1 = 0 \in \Gamma(U_j, \mathcal{G}|_{U_j})$ for some $n_j$

$\implies t_1 \otimes s^{n_j}|_{U_j} = 0 \in \Gamma(U_j, \mathcal{G} \otimes \mathcal{L}^{n_j})$

Take $n = \max\{n_1, \dots, n_r\}$. Then $t_1 \otimes s^n|_{U_j} = 0\ \forall j$

$\implies t_1 \otimes s^n = 0 \in \Gamma(Y, \mathcal{G} \otimes \mathcal{L}^n)$

(iii) Since $D_s \cap U_j = D(f_j)$ in $U_j$ and

$\mathcal{G}(D_s \cap U_j) = \mathcal{G}(U_j)[1/f_j]$ [$\because \mathcal{G}$ is qcoh]

$\exists\ n_j$ s.t $f_j^{n_j} \cdot t$ is the restriction of a section in $\Gamma(\mathcal{G}, U_j)$ to $D(f_j) = D_s \cap U_j$. This means

$\exists\ t_j \in \Gamma(U_j, \mathcal{G} \otimes \mathcal{L}^{n_j})$ s.t $t \otimes s^{n_j} = t_j|_{D_s \cap U_j}$

Let $n_0 = \max\{n_1, n_2, \dots, n_r\}$

Set $t'_j = t_j \otimes s^{n_0 - n_j} \in \Gamma(U_j, \mathcal{G} \otimes \mathcal{L}^{n_0})$

On $U_{j_1} \cap U_{j_2} \cap D_s$ $\quad t'_{j_1} = t'_{j_2} = t|_{U_{j_1} \cap U_{j_2} \cap D_s} \otimes s^{n_0}|_{U_{j_1} \cap U_{j_2} \cap D_s}$

By our hypothesis $U_{j_1} \cap U_{j_2} = \bigcup_{\alpha \in I} V_\alpha^{j_1 j_2}$, where $|I| < \infty$.

Using (ii) on each $V_\alpha^{j_1 j_2}$ and finiteness of the covering $U_{j_1} \cap U_{j_2} = \bigcup V_\alpha^{j_1 j_2}$

... $t'_{j_1} \otimes s^{n_{j_1 j_2}} = t'_{j_2} \otimes s^{n_{j_1 j_2}}$ $\quad n_0 + n_{j_1 j_2}$

by ... on each $V_\alpha^{j_1 j_2}$

Using (ii) and finiteness of the covering $U_{j_1} \cap U_{j_2} = \cup V_\alpha$

find $n_{j_1 j_2}$ such that $t'_{j_1} \otimes s^{n_{j_1 j_2}} = t'_{j_2} \otimes s^{n_{j_1 j_2}}$ $\in \Gamma(U_{j_1} \cap U_{j_2}, \mathcal{G} \otimes \mathcal{L}^{n_0 + n_{j_1 j_2}})$

set $n = \max_{j_1, j_2 \in I} \{ n_0 + n_{j_1 j_2} \}$

Then $t'_j \otimes s^n \in \Gamma(U_j, \mathcal{G} \otimes \mathcal{L}^n) \ \forall j$

and $t'_{j_1} \otimes s^n|_{U_{j_1} \cap U_{j_2}} = t'_{j_2} \otimes s^n|_{U_{j_1} \cap U_{j_2}} \ \forall j_1, j_2$

$\Rightarrow \tilde{t} \in \Gamma(Y, \mathcal{G} \otimes \mathcal{L}^n)$ s.t. $\tilde{t}|_{U_j} = t_j \otimes s^n \ \forall j$

Clearly on $\tilde{t}|_{D_s} = t \otimes s^n$ on $\Gamma(D_s, \mathcal{G} \otimes \mathcal{L}^n)$

Back to (†)

injectivity. If $s/g_i^n$ goes to zero

$s|_{D_+(g_i)} = 0 \Rightarrow \exists n_i$ s.t. $s \cdot g_i^{n_i} = 0 \in \Gamma(X, \mathcal{F}(n_i d))$

$\Rightarrow s/g_i^n = 0$ in $(\Gamma_* \mathcal{F}[1/g_i])_0$

Surjectivity. Note $X = \cup D_+(g_i)$

$D_+(g_\alpha) \cap D_+(g_\beta) = D_+(g_\alpha g_\beta)$

So we can apply (ii) of Lemma above on $X$.

Given $t \in \mathcal{F}(D_+(g_i))$. $\exists n_i \in \mathbb{N}$ s.t.

$t \otimes g_i^{n_i} = \tilde{t}|_{D_+(g_i)}$ for some $\tilde{t} \in \Gamma(X, \mathcal{F}(n_i d))$

Then $\tilde{t}/g_i^{n_i} \in (\Gamma_* \mathcal{F}[1/g_i])_0$ with its image

in $\mathcal{F}|_{D_+(g_i)}$ being $t$.

**Thm:** Assume that $\exists$ homog elts of +ve deg $g_1, g_2, \ldots g_r$ s.t. $R = R_0[g_1, g_2, \ldots g_r]$ and $\mathrm{Proj}(R) = X$ is locally noeth. Given $\mathcal{F} \in \mathrm{Coh}(X)$, $\exists$ a finitely gen $M \in \mathrm{Mod}_R^{gr}$ s.t. $\tilde{M} \xrightarrow{\sim} \mathcal{F}$

Pf. Choose $n_1, n_2, \ldots n_r$ s.t. $\deg g_i^{n_i} = \deg g_j^{n_j} = d \ \forall i, j$

Then $X = \bigcup_{i=1}^{r} D_+(g_i^{n_i})$. So $\mathcal{O}_X(d)$ is invertible.

Realize $g_i^{n_i} \in \Gamma(X, \mathcal{O}_X(d))$. Note $D_{g_i^{n_i}} = D_+(g_i^{n_i})$

↑ $g_i^{n_i}$ is thought of in $\Gamma(X, \mathcal{O}_X(d))$

Since $\mathcal{F}$ is coh, $\Gamma(D_+(g_i^{n_i}), \mathcal{F})$ is a f.g $\Gamma(D_+(g_i^{n_i}), \mathcal{O}_X)$

$\Gamma(X, \mathcal{O}_X(d))$

Since $\mathcal{F}$ is coh, $\Gamma(D_+(g_i^{n_i}), \mathcal{F})$ is a f.g $\Gamma(D_+(g_i^{n_i}), \mathcal{O}_X)$ mod. By the lemma above, $\exists\, s_1, s_2, \ldots, s_{r_i} \in \Gamma(X, \mathcal{F}(d_i d))$ such that $\{s_\alpha / g_i^{n_i d_i}\}_{\alpha = 1, \ldots, r_i}$ is a set of gen.

By varying $i$ and possibly increasing $d_i$, we can find $m \in \mathbb{N}$ & finitely many elts $t_1, t_2, \ldots, t_n \in \Gamma(X, \mathcal{F}(dm))$. s.t $\forall i$, $\{t_i / g_i^{n_i m}\}_{i = 1, \ldots, n}$ generate the $\Gamma(D_+(g_i^{n_i m}), \mathcal{O}_X)$ mod $\mathcal{F}(D_+(g_i^{n_i}))$.

Let $M$ be the (finitely generated) $R$-submodule of $\Gamma_* \mathcal{F}$ generated by $t_1, t_2, \ldots, t_n$.

Claim: The inclusion map $M \hookrightarrow \Gamma_* \mathcal{F}$ induces an isom
$$\tilde{M} \longrightarrow \widetilde{\Gamma_* \mathcal{F}} \quad \text{in } \mathrm{Mod}_{\mathcal{O}_X}.$$

<u>Pf</u> It's enough to check that for each $i$ the induced map $\Gamma(D_+(g_i^{n_i}), \tilde{M}) \longrightarrow \Gamma(D_+(g_i^{n_i}), \widetilde{\Gamma_* \mathcal{F}})$ is an isom.
$$\begin{array}{ccc} \Gamma(D_+(g_i^{n_i}), \tilde{M}) & \longrightarrow & \Gamma(D_+(g_i^{n_i}), \widetilde{\Gamma_* \mathcal{F}}) \\ \uparrow \wr & & \uparrow \wr \\ (M[1/g_i^{n_i}])_0 & \longrightarrow & (\Gamma_* \mathcal{F}[1/g_i^{n_i}])_0 \end{array}$$

The injectivity follows as $M \subseteq \Gamma_* \mathcal{F}$; surjectivity follows from the diag
$$\begin{array}{ccc} (M[1/g_i^{n_i}])_0 & \longrightarrow & \mathcal{F}(D_+(g_i^{n_i m})) \\ & \searrow \quad \circlearrowright & \nearrow \wr \\ & & (\Gamma_* \mathcal{F}[1/g_i^{n_i m}])_0 \end{array}$$
where the top arrow is surj by construction.

<u>Invertible sheaves:</u>

<u>Def:</u> $X$ be a scheme. A locally free sheaf of rk 1 is called an invertible sheaf.

<u>Prop:</u> i) Let $\mathcal{L}$ be an invertible sheaf. Then
$$\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L} \to \mathcal{O}_X \qquad (\varphi \otimes s) \mapsto \varphi(s)$$
is an isom

ii) For an $\mathcal{O}_X$-mod $\mathcal{F}$, suppose there is an $\mathcal{O}_X$-mod $\mathcal{G}$ and an isom $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{O}_X$. Then $\mathcal{F}$ is invertible.

<u>Pf</u> (i) Given $x \in X$, choose an open nbhd $U$ of $x$ s.t $\mathcal{L}|_U \xleftarrow{\sim} \mathcal{O}_U$. We have a diag using this isom

$$\begin{array}{ccccc}
\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L},\mathcal{O}_X)|_U & \otimes_{\mathcal{O}_U} & \mathcal{L}|_U & \longrightarrow & \mathcal{O}_U \\
\downarrow\wr & & \uparrow\wr & & \Vert\, id \\
\mathcal{O}_U \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_U,\mathcal{O}_U) & \otimes & \mathcal{O}_U & \longrightarrow & \mathcal{O}_U
\end{array}$$

Since the bottom row is an isom, we are done.

(ii) Check at stalks.

**Prop.** $X$ be a scheme. The isom classes of invertible $\mathcal{O}_X$-mods form an abelian group under $\otimes$ operation, denoted $\mathrm{Pic}(X)$ - called the Picard group or the group of invertible sheaf.

**Prop\Eg** - • $R$ be an $\mathbb{N}$-graded ring. Suppose $\exists\, g_1, g_2, \ldots, g_r$ each of deg $d$ such that $\mathrm{Proj}(R) = \bigcup_{i=1}^{r} D_+(g_i)$.
Then for each $n$, $\mathcal{O}_X(nd)$ is invertible.

- So if $d=1$, each $\mathcal{O}_X(n)$ invertible
- Assume $R$ is gen over $R_0$ by deg 1 elt as an alg (i.e $R$ standard graded), then $d=1$ and each $\mathcal{O}_X(n)$ is invertible.

**Eg:** We will see $\mathrm{Pic}(\mathbb{A}^n_{\mathbb{C}}) \cong \{id\}$, $\mathrm{Pic}(\mathbb{P}^n_{\mathbb{C}}) \xrightarrow{\sim} \mathbb{Z}\cdot\mathcal{O}(1)$

$\mathbb{P}^n_{\mathbb{C}} = \mathrm{Proj}(\mathbb{C}[x_0, \ldots, x_n])$

End of 13.11.24 Lecture.

**Def.** $X$ scheme, $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$ is called globally generated if there is a surjection of $\mathcal{O}_X$ mods $\bigoplus_I \mathcal{O}_X \longrightarrow \mathcal{F}$, ($|I|$ need not be finite)

**Rmk.** Note $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) \xleftarrow{\sim} \Gamma(X,\mathcal{G})$. So giving a surj morphism of $\mathcal{O}_X$-mod $\bigoplus_I \mathcal{O}_X \to \mathcal{G}$ is the same as choosing $|I|$ many elts of $\Gamma(X,\mathcal{G})$, such that those generate every stalk $\mathcal{G}_x$, $x \in X$.

**Def:** An invertible $\mathcal{O}_X$-mod $\mathcal{L}$ is called ample, if for any $\mathcal{F} \in \mathrm{Coh}(X)$ $\exists\, n_{\mathcal{F}} \in \mathbb{N}$ s.t $\forall\, n \geq n_{\mathcal{F}}$, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n$ is globally generated.

**Prop:** $\mathcal{L}$ is ample $\iff$ $\mathcal{L}^n$ is ample for some $n \in \mathbb{N}_{>0}$.

**Thm:** Let $R = \bigoplus_{\lambda \in \mathbb{N}} R_\lambda$ be a noetherian graded ring. $X = \mathrm{Proj}(R)$
If for some $m > 0$, $\mathcal{O}_X(m)$ is invertible, $\mathcal{O}_X(m)$ is ample.

Pf. Recall $R$ is noeth iff $R_0$ is noeth and $\exists\, g_1, g_2, \dots, g_r$ homo of +ve deg such that $R = R_0[g_1 \dots g_r]$

Let $d = \operatorname{lcm}(\deg g_1, \deg g_2, \dots, \deg g_r)$

Lemma: Let $d'$ be a +ve integer multiple of $d$. Then the graded subring $\bigoplus_{j=0}^{\infty} R_{d'rj} \subseteq R$ is generated as an $R_0$-algebra by elts of degree $d'r$.

Pf. See

By the Prop before it's enough to show that $\mathcal{O}_X(dmr) \cong \mathcal{O}_X(m)^{\otimes dr}$ is ample. We show this.

Recall $R^{(dmr)}$ is the graded ring whose underlying ring is $\bigoplus_{j=0}^{\infty} R_{dmrj}$ but whose $\lambda$-th graded piece is $R_{dmr\lambda}$. By the lemma above $R^{(mdr)}$ is generated over it's degree 0-piece - $R_0$, by it's set of degree 1 elts. So $R^{(mdr)}$ is a standard graded ring.

The above modifications with the following Prop often allows to reduce a problem about non-standard graded ring to a standard graded ring.

Prop: Let $S$ be an $\mathbb{N}$-graded ring, $a \in \mathbb{N}$. The inclusion map of rings $S^{(a)} \hookrightarrow S$ (it takes deg $j$ elts to deg $aj$ elts) induces an isom $\varphi: \operatorname{Proj} S \longrightarrow \operatorname{Proj}(S^{(a)})$ such that $\varphi^*(\mathcal{O}_{S^{(a)}}(1)) \cong \mathcal{O}_S(a)$

Pf: Ex.

Returning to the proof of ampleness, take $a = mdr$.

Consider $R^{(a)} \hookrightarrow R$.

Claim: If $S$ is a standard graded noeth ring. $\mathcal{O}_{\operatorname{Proj}(S)}(1)$ is ample.

Pf. Given a coh sheaf $\mathcal{G}$, choose a finitely generated $S$ mod $N$ such that $\mathcal{F} \cong \tilde{N}$. Choose homo elts $n_1, n_2, \dots, n_s$ of $N$ that generates $N$. Permuting the order assume $\deg n_1 \leq \deg n_2 \leq \dots \leq \deg n_s$

$n_1, n_2, \dots, n_s$ of $N$ that generates $N$. Permuting the order assume $\deg n_1 \le \deg n_2 \le \cdots \le \deg n_s$.

<u>Claim</u> For $\lambda \ge \lambda' \ge \deg n_s$, $N_\lambda = S_{\lambda - \lambda'} \cdot N_{\lambda'}$

<u>Pf.</u> For $\lambda \ge \deg n_s$, choose $x \in N_\lambda$.

$x = f_1 n_1 + \cdots + f_s n_s$, $\deg f_i = \deg x - \deg n_i$

So $f_i$ is sum of pdt of $\deg x - \deg n_i$ monomials in the deg 1 generators of $S$. So each $f_i$ can be written as $f_i = \sum \tilde{f}^i_j \cdot q_j$, where $\deg \tilde{f}^i_j = \deg n_s - \deg n_i$, $\deg q_j = \deg x - \deg n_s$. Thus $x = \sum_i \sum_j (\tilde{f}^i_j \cdot n_i) \cdot q_j$

$\deg \tilde{f}^i_j \cdot n_i = \deg n_s \quad \forall i, j$. So $N_\lambda = S_{\lambda - \deg n_s} \cdot N_{\deg n_s}$

So $N_\lambda = S_{\lambda - \lambda'} \cdot S_{\lambda' - \deg n_s} N_{\deg n_s}$

$= S_{\lambda - \lambda'} \cdot N_{\lambda'}$

<u>Claim.</u> For $\lambda \ge \deg n_s$, $\tilde{N}(\lambda)$ is glob gen.

<u>Pf.</u> Take $\lambda \ge \deg n_s$.

- Since $S$ is noeth, $N$ f.g, $\exists$ homo elts $\alpha_1, \alpha_2, \dots, \alpha_t$ generating $N_\lambda$ as an $S_0$-mod.

Consider the graded map

$$(*) \quad \bigoplus^t S \longrightarrow N(\lambda)$$
$$1 \longmapsto \alpha_i$$

By the claim above the map is surj onto $\bigoplus_{j \ge \lambda} N_j$.

So the cokernel is annihilated by $(S_+)^{\deg s}$.

So the sheaf given by cokernel on $\operatorname{Proj} S$ is $0$.

Thus taking $\sim$ in $*$, get a surj map

$$\bigoplus^t \mathcal{O}_{\operatorname{Proj}(S)} \longrightarrow \widetilde{N(\lambda)} \longrightarrow 0$$

<u>Claim.</u> Since $S$ is standard graded $\widetilde{N(\lambda)} \xleftarrow{\sim} \tilde{N} \otimes \mathcal{O}_{\operatorname{Proj}(S)}(\lambda)$

<u>Pf.</u> Ex.

<u>Recall</u> $a = \dim R$ and $R^{(a)} \hookrightarrow R$ induces an isom

$\varphi : \operatorname{Proj}(R) \longrightarrow \operatorname{Proj}(R^{(a)})$ such that $\varphi^*(\mathcal{O}_{R^{(a)}}(1))$

$= \mathcal{O}_R(a)$

Since $\varphi$ is an isom, pull back of an ample sheaf is ample. Thus $\mathcal{O}_R(dm+e)$ is ample $\Rightarrow$ $\mathcal{O}(d)$ is ample.